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Hitting of a line by two-dimensional symmetric stable Lévy processes: an approach based on modified resolvents

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Abstract

Let $(X(t), Y(t))$ be a symmetric α-stable Lévy process on $\mathbb{R}^2$ with $1 < \alpha \leq 2$. We announce a multivariate asymptotic estimate involving the first hitting time/place of a half-line. We deduce explicitly the density of the first hitting distribution of a line. The method is based on some modified version of quantities in the celebrated potential theory. We also discuss properties of quantities arising in our modification.

1 Introduction and the result

In [4] and [5], the author studied trivariate asymptotic estimates involving the first hitting time of the nonnegative-half of the first axis, the first hitting place thereon, and the sojourn time on the first axis up to then, by a random walk and a Brownian motion, respectively.

Note that more precise information can be retrieved from this kind of estimates such as the tail probability concerning both the first hitting time and place, than from the tail probability of the first hitting time.

Let $1 < \alpha \leq 2$. In this note, we are mainly concerned with the α-stable Lévy process $(X(t), Y(t))$ with rotational symmetry on $\mathbb{R}^2$ starting from $(x_0, y_0) \in \mathbb{R}^2$. Its law and expectation are denoted by $P_{(x_0,y_0)}$ and $E_{(x_0,y_0)}$, respectively, and are determined by $E_{(0,0)}[e^{i\xi_1 X(t)+i\xi_2 Y(t)}] = e^{-t(\xi_1^2+\xi_2^2)^{\alpha/2}}$ for $(\xi_1, \xi_2) \in \mathbb{R}^2$. Let $L_Y(t)$ be the local time at 0 for $Y(\cdot)$: $L_Y(t) = \lim_{\epsilon\to+0} \frac{1}{2\epsilon}\int_0^t 1_{(-\epsilon,\epsilon)}(Y(s))ds$.

For $a \in \mathbb{R}$, we set

$$\tau(a) = \inf\{t \geq 0 | Y(t) = 0, X(t) \geq a\}. \tag{1.1}$$

We also set $\Phi_\alpha(\xi_1, \mu_2) = 2\pi / \int_{\mathbb{R}} \frac{d\xi_2}{\mu_2+(\xi_1^2+\xi_2^2)^{\alpha/2}}$, $C_1(\alpha) = \Phi_\alpha(1,0) = 2\pi/B(\frac{1}{2}, \frac{\alpha-1}{2})$, $C_2(\alpha) = \Phi_\alpha(0,1) = \alpha \sin\frac{\pi}{\alpha}$, and

$$I_\alpha(\mu_0, \mu_1, \mu_2) = \int_{-\infty}^{\infty} \frac{dt}{2\pi(t^2+1)} \log(\mu_0 + \Phi_\alpha(\mu_1 t, \mu_2)) \tag{1.2}$$

for $\xi_1 \in \mathbb{R}$ and $\mu_i \geq 0$ $(i = 0, 1, 2)$ such that $\mu_0 + \mu_1 + \mu_2 > 0$.

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To state the main theorem, we introduce a family of holomorphic functions. Let $\mathbb{C}_+ = \{z \in \mathbb{C} | \Im z > 0\}$, $\overline{\mathbb{C}_+} = \{z \in \mathbb{C} | \Im z \geq 0\}$ and set

$$\varphi_\alpha(z;\mu_0,\mu_2) = \exp\left(\frac{-1}{2\pi i}\int_{-\infty}^{\infty}\frac{z}{t^2-z^2}\log(\mu_0+\Phi_\alpha(t,\mu_2))dt\right) \tag{1.3}$$

for $z \in \mathbb{C}_+$ and $\mu_i \geq 0$ $(i = 0,2)$ such that $\mu_0 + \mu_2 > 0$. We can extend $\varphi_\alpha(z;\mu_0,\mu_2)$ for $z \in \mathbb{R}$ by continuity. We also set

$$\varphi_\alpha(z;0,0) = \frac{1}{\sqrt{C_1(\alpha)}}(-iz)^{-(\alpha-1)/2} \quad \text{for } z \in \overline{\mathbb{C}_+}\setminus\{0\},$$

where we employ the branch such that $1^{-(\alpha-1)/2} = 1$.

Theorem 1.1 *Let* $a > 0$, $\mu_i \geq 0$ $(i = 0,1,2)$, *and* $\mu_0 + \mu_1 + \mu_2 > 0$.

(i) *It holds*

$$\begin{aligned}&E_{(-a,0)}\left[e^{-\mu_0 L_Y(\tau(0))-\mu_1 X(\tau(0))-\mu_2\tau(0)}\right]\\ &= e^{\mu_1 a} - \frac{e^{\mu_1 a}}{\varphi_\alpha(i\mu_1;\mu_0,\mu_2)}\int_{-\infty}^{\infty}d\theta\frac{1-e^{-ia(\theta-i\mu_1)}}{2\pi i(\theta-i\mu_1)}\varphi_\alpha(\theta;\mu_0,\mu_2).\end{aligned}$$

(ii) *As* $s \to +0$,

$$\begin{aligned}&1 - E_{(-a,0)}\left[e^{-\mu_0 s^{2(\alpha-1)}L_Y(\tau(0))-\mu_1 s^2 X(\tau(0))-\mu_2 s^{2\alpha}\tau(0)}\right]\\ &\sim \frac{\exp(I_\alpha(\mu_0,\mu_1,\mu_2))a^{(\alpha-1)/2}}{\sqrt{C_1(\alpha)}\Gamma(1+\frac{\alpha-1}{2})}s^{\alpha-1},\end{aligned}$$

where $\sim$ *means that the ratio of the both sides converges to* 1.

Since the method of proof applies to symmetric α-stable Lévy processes on $\mathbb{R}^2$, we restate the theorem for such processes in the forthcoming paper [6]. Here $(X(t), Y(t))$ is symmetric iff $(X(t)-X(0), Y(t)-Y(0))$ has the same law as $(X(0)-X(t), Y(0)-Y(t))$. In Section 4, we give a generalization of the theorem for such $(X(t), Y(t))$ that $X(t)$ and $Y(t)$ are independent, $X(t)$ is symmetric β-stable, and $Y(t)$ is symmetric α-stable.

We could not calculate explicitly the definite integral $I_\alpha(\mu_0,\mu_1,\mu_2)$ defined in (1.2) but some marginal values can be evaluated, e.g., $\exp(I_\alpha(0,\mu_1,0)) = \sqrt{C_1(\alpha)}\mu_1^{(\alpha-1)/2}$ and $\exp(I_\alpha(0,0,\mu_2)) = \sqrt{C_2(\alpha)}\mu_2^{(\alpha-1)/\alpha}$.

It is elementary to obtain the following corollary by a Tauberian theorem, the strong Markov property, and Theorem 1.2(i) below.

Corollary 1.1 (i) *We have*

$$P_{(-a,0)}[\tau(0) > A] \sim \frac{\sqrt{C_2(\alpha)}a^{(\alpha-1)/2}}{\sqrt{C_1(\alpha)}\Gamma(1+\frac{\alpha-1}{2})\Gamma(1-\frac{\alpha-1}{2\alpha})}A^{-\frac{(\alpha-1)}{2\alpha}} \quad \textit{as } A \to +\infty,$$

where $C_1(\alpha) = 2\pi / B(\frac{1}{2}, \frac{\alpha-1}{2})$ *and* $C_2(\alpha) = \alpha \sin\frac{\pi}{\alpha}$.

(ii) *If* $y_0 \neq 0$ *and* $x_0 \in \mathbb{R}$, *we have, as* $s \to +0$,

$$1 - E_{(x_0,y_0)}\left[e^{-\mu_0 s^{2(\alpha-1)} L_Y(\tau(0)) - \mu_1 s^2 X(\tau(0)) - \mu_2 s^{2\alpha}\tau(0)} \right]$$
$$\sim \ s^{\alpha-1} \frac{\exp(I_\alpha(\mu_0,\mu_1,\mu_2))}{\sqrt{C_1(\alpha)}\Gamma(1+\frac{\alpha-1}{2})} \int_{-\infty}^{-x_0/|y_0|} \frac{(1+t^2)^{-\alpha/2}}{B(\frac{1}{2},\frac{\alpha-1}{2})} \,|\, x_0 + |y_0| t \,|^{(\alpha-1)/2}\, dt.$$

In the course of the proof of Theorem 1.1, we obtain an explicit formula for the first hitting distribution of a line.

The law of a Lévy process $(X(t), Y(t))$ on $\mathbb{R}^2$ is determined by the characteristic exponent $\Psi(\xi_1,\xi_2)$ satisfying $E_{(0,0)}[e^{i\xi_1 X(t) + i\xi_2 Y(t)}] = e^{-t\Psi(\xi_1,\xi_2)}$ for $(\xi_1,\xi_2) \in \mathbb{R}^2$. If $(X(t), Y(t))$ is symmetric in the sense that $(X(t) - X(0), Y(t) - Y(0))$ has the same law as $(X(0) - X(t), Y(0) - Y(t))$, we have $\Psi(\xi_1,\xi_2) = \Psi(-\xi_1,-\xi_2)$ and hence Ψ is real-valued.

Theorem 1.2 *Set* $T_0^Y := \inf\{t \geq 0 | Y(t) = 0\}$.

(i) *Let* $(X(t), Y(t))$ *be an* α*-stable Lévy process with rotational symmetry on* $\mathbb{R}^2$ *and* $C_{\alpha,\mathrm{rot}}$ *be a real random variable such that* $P[C_{\alpha,\mathrm{rot}} \in dx] = B(\frac{1}{2},\frac{\alpha-1}{2})^{-1}(1+x^2)^{-\alpha/2}dx$.

Then $P_{(x_0,y_0)}[X(T_0^Y) \in dx] = P[y_0 C_{\alpha,\mathrm{rot}} + x_0 \in dx]$.

(ii) *More generally, if* $(X(t), Y(t))$ *is a genuinely two-dimensional symmetric* α*-stable Lévy process such that* $E_{(0,0)}[e^{i\xi_1 X(t) + i\xi_2 Y(t)}] = e^{-t\Psi(\xi_1,\xi_2)}$, *set* $P[C_\Psi \in dx] = \dfrac{\Psi(1,x)^{-1}dx}{\int_{\mathbb{R}} \Psi(1,t)^{-1}dt}$.

Then $P_{(x_0,y_0)}[X(T_0^Y) \in dx] = P[y_0 C_\Psi + x_0 \in dx]$.

The proof is given in Section 2 by an approach based on modified resolvents. We characterize some quantities related to modified resolvents in Section 5.

In Section 2, we also study the hitting times of two parallel lines and some formula concerning the last exit time from a line.

To our knowledge, there are only two papers in the literature concerning explicit hitting distribution of sets by multidimensional stable Lévy processes. [2] obtained the first hitting distribution of $\{x \in \mathbb{R}^d | |x| > r\}$ and $\{x \in \mathbb{R}^d | |x| < r\}$, and [7] obtained that of $\{x \in \mathbb{R}^d | |x| = r\}$, by an α-stable Lévy processes with rotational symmetry. Theorem 1.2 is restricted to the case for dimension 2, but needs not the rotational symmetry. Unfortunately, the author has not succeeded in extending our result to the case for dimension 3 or higher.

It seems interesting to compare Theorem 1.2 with the formula (5.12) in [8], which concentrates on the one-dimensional symmetric α-stable Lévy process. Let $X(t)$ and $Y(t)$ are independent symmetric α-stable Lévy processes with $1 < \alpha \leq 2$ and $P[C_\alpha \in dx] = \frac{\alpha}{2\pi}\sin(\frac{\pi}{\alpha})(1+|x|^\alpha)^{-1}dx$. Then it is shown that $P_{(x_0,y_0)}[X(T_0^Y) \in dx] = P[y_0 C_\alpha + x_0 \in dx]$. Our Theorem 1.2(ii) contains this formula: in this case we have $\Psi(\xi_1,\xi_2) = |\xi_1|^\alpha + |\xi_2|^\alpha$ and $P[C_\Psi \in dx] = P[C_\alpha \in dx]$. The variable C_α is called an α-Cauchy variable in [8] since its law reduces to the Cauchy distribution if $\alpha = 2$.

Let us also remark that Theorem 1.2(i) and [8, (5.12)] are different stable-analogs of the hitting distribution of a line by a two-dimensional standard Brownian motion, namely the Cauchy distribution. A two-dimensional standard Brownian motion has the independent components and is of rotational symmetry. But a two-dimensional symmetric α-stable Lévy process does not have these two properties at the same time. [8] retains independence of components while Theorem 1.2(i) is based on rotational symmetry. We may consider $\mathcal{C}_{\alpha,\mathrm{rot}}$ as another α-Cauchy variable.

2 Modified resolvents and proof of Theorem 1.2

In this section, we introduce the modified resolvents $U(dy;\xi_1,\mu)$ and its density $u(y;\xi_1,\mu)$ and apply them to determine the joint law of the first hitting time and place of a line. The resolvents $U(dy;\xi_1,\mu)$ are modified ones in the sense that they reduce, if $\xi_1=0$, to μ-resolvents for a one-dimensional Lévy process $Y(t)$ as in [1, §I.2].

Let $(X(t),Y(t))$ be a two-dimensional Lévy process starting from $(x_0,y_0)\in\mathbb{R}^2$. Its law and expectation are denoted by $P_{(x_0,y_0)}$ and $E_{(x_0,y_0)}$, respectively. Let $\mathcal{F}_t$ be the $P_{(x_0,y_0)}$-completion of $\sigma((X(s),Y(s));s\in[0,t])$. We denote its characteristic exponent by $\Psi(\xi_1,\xi_2)$, i.e. it holds $E_{(0,0)}[e^{i\xi_1X(t)+i\xi_2Y(t)}]=e^{-t\Psi(\xi_1,\xi_2)}$ for $(\xi_1,\xi_2)\in\mathbb{R}^2$.

Assume $\Psi(\xi_1,\xi_2)$ satisfies

$$\int_{\mathbb{R}}\left|\frac{1}{1+\Psi(0,\xi_2)}\right|d\xi_2<\infty. \tag{2.1}$$

Then it is well-known(see [1, Corollary II.20, Theorem V.1, and Proposition V.2]) that $Y(t)$ admits a local time process $L_Y(y,t)=\lim_{\varepsilon\to+0}\frac{1}{2\varepsilon}\int_0^t 1_{\{|Y(s)-y|<\varepsilon\}}ds$ and $t\mapsto L_Y(y,t)$ is a.s. continuous.

Note that (2.1) is a bit stronger than the existence of such $L_Y(y,t)$: (2.1) implies that $\int_{\mathbb{R}}\Re\frac{1}{1+\Psi(0,\xi_2)}d\xi_2<\infty$ and a single point is regular for itself for $Y(t)$; these conditions are sufficient for the existence of $L_Y(y,t)$ as above. We assume (2.1) since it facilitates (2.5) below and the Lévy processes of our interest, such as symmetric α-stable processes with $1<\alpha\le 2$, satisfy (2.1).

One can show that, for any bounded Borel function $f(y)$ on $\mathbb{R}$,

$$\int_0^u e^{i\xi_1X(t)}f(Y(t))dt=\int_{\mathbb{R}}dy\,f(y)\int_0^u e^{i\xi_1X(t)}d_tL_Y(y,t) \tag{2.2}$$

by standard arguments. Set

$$\begin{aligned} U(dy;\xi_1,\mu) &:= E_{(0,0)}\left[\int_0^\infty e^{i\xi_1X(t)-\mu t}1_{\{Y(t)\in dy\}}dt\right], &(2.3)\\ u(y;\xi_1,\mu) &:= E_{(0,0)}\left[\int_0^\infty e^{i\xi_1X(t)-\mu t}d_tL_Y(y,t)\right] &(2.4)\end{aligned}$$

for $\xi_1,y\in\mathbb{R}$ and $\mu>0$.

Note that these quantities correspond to the following ones in [1] if $\xi_1 = 0$: (2.2) reduces to $\int_0^u f(Y(t))dt = \int_{\mathbb{R}} dy\, f(y) L_Y(y,u)$ in [1, §V.1]; $U(dy;0,\mu)$ is the μ-resolvent $U^\mu(0,dy)$ for $Y(t)$ in [1, §I.2]; and then

$$u(y;0,\mu) = E_{(0,0)}\left[\int_0^\infty e^{-\mu t} d_t L_Y(y,t)\right] = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-iy\xi_2}}{\mu + \Psi(0,\xi_2)} d\xi_2$$

is the continuous version of the density for $U^\mu(0,dy)$ in [1, §II.5]. In Section 5, we discuss their properties from the potential theoretic viewpoint.

Lemma 2.1 *Assume* (2.1).

(i) *The function* $y \mapsto u(y;\xi_1,\mu)$ *is a version of the density for* $U(dy;\xi_1,\mu)$.

(ii) *Assume* $\xi_2 \mapsto 1/|\mu + \Psi(\xi_1,\xi_2)|$ *is integrable for any fixed* ξ_1. *Then we have*

$$u(y;\xi_1,\mu) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-iy\xi_2}}{\mu + \Psi(\xi_1,\xi_2)} d\xi_2. \tag{2.5}$$

Proof. We refer the reader to the forthcoming paper [6] for the proof. □

Note that if $1/|\mu + \Psi(\xi_1,\xi_2)|$ is integrable for some $\mu > 0$, then it is integrable for any $\mu > 0$.

Note also that $1/|\mu + \Psi(\xi_1,\xi_2)|$ is integrable if the process is genuinely two-dimensional and $\forall c > 0, \forall(\xi_1,\xi_2)$, $\Psi(c\xi_1, c\xi_2) = c^\alpha \Psi(\xi_1,\xi_2)$. Indeed, $\Re\Psi(\xi_1,\xi_2) \geq 0$, Ψ vanishes only at $(0,0)$, and we have $1/|\mu + \Psi(\xi_1,\xi_2)| \sim |\xi_2|^{-\alpha}/|\Psi(\xi_1/\xi_2, 1)| \sim |\xi_2|^{-\alpha}/|\Psi(0,1)|$ as $\xi_2 \to \infty$. A similar bound holds when $\xi_2 \to -\infty$.

We next set, for any fixed $\xi_1 \in \mathbb{R}$ and $\mu > 0$,

$$N(t) = e^{i\xi_1 X(t) - \mu t} u(-Y(t);\xi_1,\mu). \tag{2.6}$$

This process is bounded since $|u(y;\xi_1,\mu)| \leq u(0;0,\mu)$ by (2.4).

Lemma 2.2 *Assume* (2.1). *Then for any starting point* $(x_0,y_0) \in \mathbb{R}^2$, *under* $P_{(x_0,y_0)}$,

(i) $N(t) + \int_0^t e^{i\xi_1 X(s) - \mu s} d_s L_Y(0,s)$ *is a u.i. martingale;*

(ii) $M(t) = e^{(1/u(0;\xi_1,\mu))L_Y(0,t)} N(t)$ *is a local martingale.*

Proof. We refer the reader to the forthcoming paper [6] for the proof. □

Let

$$L_Y^{-1}(t) := \inf\{s \geq 0 | L_Y(0,s) > t\} \quad \text{and} \quad \Xi(t) = X(L_Y^{-1}(t)). \tag{2.7}$$

Then, under $P_{(x_0,0)}$, $(\Xi(t), L_Y^{-1}(t))$ is a two-dimensional Lévy process starting from $(x_0, 0)$.

Lemma 2.3 *Assume* (2.1) *and the condition in Lemma 2.1(ii).*

Then the Lévy process $(\Xi(t), L_Y^{-1}(t))$ *has the following Fourier-Laplace characteristic exponent:* $E_{(0,0)}[e^{i\xi_1\Xi(t) - \mu L_Y^{-1}(t)}] = e^{-t\Phi(\xi_1,\mu)}$ *with*

$$\Phi(\xi_1,\mu) = 2\pi \Big/ \int_{\mathbb{R}} \frac{1}{\mu + \Psi(\xi_1,\xi_2)} d\xi_2 \quad \text{for } \xi_1 \in \mathbb{R} \text{ and } \mu > 0. \tag{2.8}$$

If $(X(t),Y(t))$ *is a genuinely two-dimensional symmetric* α*-stable Lévy process,* (2.8) *is also valid for* $\xi_1 \neq 0$ *and* $\mu = 0$.

Proof. If $\mu > 0$, we stop $M(t)$ at $L_Y^{-1}(t)$ to obtain a bounded martingale. Then we have $E_{(0,0)}\left[e^{(1/u(0;\xi_1,\mu))t}e^{-\mu L_Y^{-1}(t)+i\xi_1 X(L_Y^{-1}(t))}u(0;\xi_1,\mu)\right] = M(0) = u(0;\xi_1,\mu)$, which implies (2.8) by (2.5).

Fix $\xi_1 \neq 0$. If $(X(t),Y(t))$ is a genuinely two-dimensional symmetric α-stable Lévy process, we have $\inf_{\xi_2\in\mathbb{R}} \Psi(\xi_1,\xi_2) > 0$ and $\Psi(\xi_1,\xi_2) \sim |\xi_2|^\alpha \Psi(0,1)$ as $|\xi_2| \to \infty$. The condition in Lemma 2.1(ii) is satisfied, as is seen in the arguments following the proof of Lemma 2.1. On one hand, we have $\lim_{\mu\to+0}\Phi(\xi_1,\mu) = 2\pi \Big/ \int_{\mathbb{R}} \frac{1}{\Psi(\xi_1,\xi_2)}d\xi_2$ by the dominated convergence. On the other hand, $E_{(0,0)}[e^{i\xi_1\Xi(t)}] = \lim_{\mu\to+0} E_{(0,0)}[e^{i\xi_1\Xi(t)-\mu L_Y^{-1}(t)}] = \exp\left(-t\lim_{\mu\to+0}\Phi(\xi_1,\mu)\right)$. □

Proof of Theorem 1.2. Fix $\xi_1 > 0$. By the same argument as the proof of Lemma 2.3, we have

$$u(y;\xi_1,0) := \lim_{\mu\to+0} u(y;\xi_1,\mu) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-iy\xi_2}}{\Psi(\xi_1,\xi_2)}d\xi_2.$$

Since $\xi_1 \neq 0$, we have $\Psi(\xi_1,\xi_2) > 0$ for any $\xi_2 \in \mathbb{R}$, and then $u(0;\xi_1,0) \in (0,\infty)$. Stopping $M(t)$ at T_0^Y, we have

$$E_{(x_0,y_0)}\left[e^{i\xi_1 X(T_0^Y)-\mu T_0^Y}\right] = \frac{e^{i\xi_1 x_0}u(-y_0;\xi_1,\mu)}{u(0;\xi_1,\mu)}.$$

We then let $\mu \to +0$ to obtain

$$E_{(x_0,y_0)}\left[e^{i\xi_1 X(T_0^Y)}\right] = \frac{e^{i\xi_1 x_0}u(-y_0;\xi_1,0)}{u(0;\xi_1,0)}. \tag{2.9}$$

By substituting $\xi_2 = \xi_1 x$, we have

$$u(y;\xi_1,0) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-iy\xi_1 x}}{\Psi(\xi_1,\xi_1 x)}\xi_1 dx = \frac{\xi_1^{1-\alpha}}{2\pi}\int_{\mathbb{R}} \frac{e^{-iy\xi_1 x}}{\Psi(1,x)}dx$$

since $\Psi(c\xi_1,c\xi_2) = c^\alpha\Psi(\xi_1,\xi_2)$. Putting this into (2.9), we have

$$\begin{aligned} E_{(x_0,y_0)}\left[e^{i\xi_1 X(T_0^Y)}\right] &= \left(\int_{\mathbb{R}} \frac{1}{\Psi(1,t)}dt\right)^{-1} \int_{\mathbb{R}} \frac{e^{i\xi_1 y_0 x + i\xi_1 x_0}}{\Psi(1,x)}dx \\ &= \int_{\mathbb{R}} e^{i\xi_1(y_0 x + x_0)} \left(\int_{\mathbb{R}} \frac{1}{\Psi(1,t)}dt\right)^{-1} \Psi(1,x)^{-1}dx. \end{aligned}$$

The comlex conjugate of both sides yields the same formula for $\xi_1 < 0$.

Then the right hand side is equal to $E[\exp(i\xi_1(y_0 C_\Psi + x_0))]$, where $P[C_\Psi \in dx] = \left(\int_{\mathbb{R}} \Psi(1,t)^{-1}dt\right)^{-1} \Psi(1,x)^{-1}dx$. □

2.1 Appendix to Section 2: hitting of a line or two parallel lines

We determine the joint law of the first hitting time/place of a line or lines. We do not need the content of this subsection in proving Theorem 1.2.

For any $a, b \in \mathbb{R}$ such that $a \neq b$, set

$$\begin{aligned} T_a^Y &= \inf\{t \geq 0 | Y(t) = a\}, \\ T_{a,b}^Y &= \inf\{t \geq 0 | Y(t) \in \{a, b\}\}. \end{aligned}$$

These are respectively the first hitting times of a line and two parallel lines.

The hitting time T_a^Y can be decomposed at the last exit time from the line $\{Y = Y(0)\}$:

$$G_a^Y = \inf\left\{t \leq T_a^Y | Y(t) = Y(0)\right\}$$

is independent of $T_a^Y - G_a^Y$.

In the following lemma, (i) is an extension of a well-known fact, see e.g. Corollary II.18 in [1]. Moreover, (ii), (iii) and (iv) are extensions of Proposition 5.4, 5.5, and Theorem 5.8 in [8], respectively.

Lemma 2.4 *Assume* (2.1) *and let* $\xi_1 \in \mathbb{R}$, $\mu > 0$, $a \neq 0$, $b \neq 0$, *and* $a \neq b$. *Then*

(i) *it holds* $E_{(0,0)}\left[e^{i\xi_1 X(T_a^Y) - \mu T_a^Y}\right] = \frac{u(a;\xi_1,\mu)}{u(0;\xi_1,\mu)}$;

(ii) *it holds*

$$\begin{aligned} &E_{(0,0)}\left[e^{i\xi_1 X(T_{a,b}^Y) - \mu T_{a,b}^Y}\right] \\ &= \frac{(u(0;\xi_1,\mu) - u(b-a;\xi_1,\mu))u(a;\xi_1,\mu) + (u(0;\xi_1,\mu) - u(a-b;\xi_1,\mu))u(b;\xi_1,\mu)}{u(0;\xi_1,\mu)^2 - u(a-b;\xi_1,\mu)u(b-a;\xi_1,\mu)}; \end{aligned}$$

if $(X(t), -Y(t)) \stackrel{\text{law}}{=} (X(t), Y(t))$ *then*

$$E_{(0,0)}\left[e^{i\xi_1 X(T_{a,b}^Y) - \mu T_{a,b}^Y}\right] = \frac{u(a;\xi_1,\mu) + u(b;\xi_1,\mu)}{u(0;\xi_1,\mu) + u(b-a;\xi_1,\mu)};$$

(iii) *it holds*

$$E_{(0,0)}\left[e^{i\xi_1 X(T_b^Y) - \mu T_b^Y}; T_b^Y < T_a^Y\right] = \frac{-u(b-a;\xi_1,\mu)u(a;\xi_1,\mu) + u(0;\xi_1,\mu)u(b;\xi_1,\mu)}{u(0;\xi_1,\mu)^2 - u(a-b;\xi_1,\mu)u(b-a;\xi_1,\mu)};$$

(iv) *it holds, with* $h^{(\alpha)}(a) = \frac{|a|^{\alpha-1}}{2\Gamma(\alpha)\sin\frac{(\alpha-1)\pi}{2}}$,

$$\begin{aligned} E_{(0,0)}\left[e^{i\xi_1 X(G_a^Y) - \mu G_a^Y}\right] &= \frac{u(0;\xi_1,\mu)^2 - u(a;\xi_1,\mu)u(-a;\xi_1,\mu)}{2h^{(\alpha)}(a)\Psi(0,1)^{-1}u(0;\xi_1,\mu)}, \\ E_{(0,0)}\left[e^{i\xi_1(X(T_a^Y) - X(G_a^Y)) - \mu(T_a^Y - G_a^Y)}\right] &= \frac{2h^{(\alpha)}(a)\Psi(0,1)^{-1}u(a;\xi_1,\mu)}{u(0;\xi_1,\mu)^2 - u(a;\xi_1,\mu)u(-a;\xi_1,\mu)}. \end{aligned}$$

Proof. Let our process start from $(0, -a)$. We stop $M(t)$ at T_0^Y. Since $L_Y(0, T_0^Y) = 0$,

$$E_{(0,-a)}\left[e^{i\xi_1 X(T_0^Y) - \mu T_0^Y}\right] = \frac{u(a;\xi_1,\mu)}{u(0;\xi_1,\mu)}.$$

By the translation invariance, we have the statement of (i).

(ii) Let c_a and c_b be such that

$$\begin{aligned} 1 &= c_a u(0;\xi_1,\mu) + c_b u(b-a;\xi_1,\mu), \\ 1 &= c_a u(a-b;\xi_1,\mu) + c_b u(0;\xi_1,\mu). \end{aligned}$$

As a corollary to (i) we have $|u(y;\xi_1,\mu)| < |u(0;\xi_1,\mu)|$ for any $y \neq 0$, $\xi_1 \in \mathbb{R}$, and $\mu > 0$, which ensures that the solution (c_a, c_b) exist:

$$\begin{aligned} c_a &= \frac{u(0;\xi_1,\mu) - u(a-b;\xi_1,\mu)}{u(0;\xi_1,\mu)^2 - u(a-b;\xi_1,\mu)u(b-a;\xi_1,\mu)}, \\ c_b &= \frac{u(0;\xi_1,\mu) - u(b-a;\xi_1,\mu)}{u(0;\xi_1,\mu)^2 - u(a-b;\xi_1,\mu)u(b-a;\xi_1,\mu)}. \end{aligned}$$

We define

$$M_{a,b}(t) = e^{i\xi_1 X(t) - \mu t}\left(c_a u(a - Y(t);\xi_1,\mu) + c_b u(b - Y(t);\xi_1,\mu)\right).$$

Then $M_{a,b}(t \wedge T^Y_{a,b})$ is a bounded martingale. Now the statement in (ii) is equivalent to $E_{(0,0)}\left[e^{i\xi_1 X(T^Y_{a,b}) - \mu T^Y_{a,b}}\right] = E_{(0,0)}\left[M(T^Y_{a,b})\right] = M_{a,b}(0) = c_a u(a;\xi_1,\mu) + c_b u(b;\xi_1,\mu)$.

If we put the symmetry assumption in (ii), $u(y;\xi_1,\mu) = u(-y;\xi_1,\mu)$ and hence $c_a = c_b = 1/\left(u(0;\xi_1,\mu) + u(b-a;\xi_1,\mu)\right)$.

(iii) Let c_a and c_b be such that

$$\begin{aligned} 0 &= c_a u(0;\xi_1,\mu) + c_b u(b-a;\xi_1,\mu), \\ 1 &= c_a u(a-b;\xi_1,\mu) + c_b u(0;\xi_1,\mu). \end{aligned}$$

Then

$$\begin{aligned} c_a &= \frac{-u(b-a;\xi_1,\mu)}{u(0;\xi_1,\mu)^2 - u(a-b;\xi_1,\mu)u(b-a;\xi_1,\mu)}, \\ c_b &= \frac{u(0;\xi_1,\mu)}{u(0;\xi_1,\mu)^2 - u(a-b;\xi_1,\mu)u(b-a;\xi_1,\mu)}. \end{aligned}$$

We define

$$N_{a,b}(t) = e^{i\xi_1 X(t) - \mu t}\left(c_a u(a - Y(t);\xi_1,\mu) + c_b u(b - Y(t);\xi_1,\mu)\right)$$

so that $N_{a,b}(t \wedge T^Y_{a,b})$ is another bounded martingale. Finally,

$$\begin{aligned} & E_{(0,0)}\left[e^{i\xi_1 X(T^Y_b) - \mu T^Y_b} ; T^Y_b < T^Y_a\right] \\ &= E_{(0,0)}\left[e^{i\xi_1 X(T^Y_{a,b}) - \mu T^Y_{a,b}} ; Y(T^Y_{a,b}) = b\right] \\ &= E_{(0,0)}\left[N(T^Y_{a,b})\right] \\ &= N_{a,b}(0) = c_a u(a;\xi_1,\mu) + c_b u(b;\xi_1,\mu). \end{aligned}$$

(iv) Recall that we normalize the local time of $Y(\cdot)$ at 0 by

$$E_{(0,0)}\left[\int_0^\infty e^{i\xi_1 X(t)-\mu t} d_t L_Y(0,t)\right] = u(0;\xi_1,\mu) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{\mu+\Psi(\xi_1,\xi_2)} d\xi_2.$$

Let us introduce Ito's excursion measure (see standard texbooks; we adopt the notations in [8, §3]). Let $\mathbb{D} = \mathbb{D}([0,\infty);\mathbb{R}^2)$ be the space of $\mathbb{R}^2$-valued cádlág paths equipped with the Skorohod topology. We define the random set

$$D := \{l > 0 | L_Y^{-1}(l) > L_Y^{-1}(l-)\}$$

and a point function $\mathbf{p}(l) \in \mathbb{D}$ on D by

$$\mathbf{p}(l)(t) := \begin{cases} (X(t+L_Y^{-1}(l-)), Y(t+L_Y^{-1}(l-))), & \text{if } t \in [0, L_Y^{-1}(l) - L_Y^{-1}(l-)), \\ (X(L_Y^{-1}(l)), Y(L_Y^{-1}(l))), & \text{otherwise.} \end{cases}$$

Remark. $Y(L_Y^{-1}(l)) = 0$ but $X(L_Y^{-1}(l))$ needs not to be 0 since X is 'running freely.'

Then it is well-known that $\mathbf{p}(\cdot)$ is a Poisson point process. Ito's excursion measure is defined as follows: if $U \in \mathcal{B}(\mathbb{D})$ and $D_U := \{l \in D | \mathbf{p}(l) \in U\}$, set

$$\mathbf{n}^\Psi[U] := E_{(0,0)}\left[\#\left(D_U \cap (0,1]\right)\right].$$

The formula

$$E_{(0,0)}[e^{i\xi_1 X(L_Y^{-1}(t))-\mu L_Y^{-1}(t)}] = e^{-t/u(0;\xi_1,\mu)}$$

as in Lemma 2.3 implies that

$$\mathbf{n}^\Psi\left[1-\exp(i\xi_1 u_1(\zeta)-\mu\zeta)\right] = 1/u(0;\xi_1,\mu), \tag{2.10}$$

where ζ is the lifetime of a generic excursion $u(\cdot) = (u_1(\cdot), u_2(\cdot)) \in \mathbb{D}$:

$$\zeta = \zeta(u) := \sup\{t \geq 0 | u_2(t) = 0\}.$$

Note that $u_1(t)$ needs not to end at 0.

Let $T_a(u_2)$ be the first hitting time of $a \in \mathbb{R}$ by the second component $u_2(\cdot)$ of a generic ($\mathbb{R}^2$-valued) excursion $u \in \mathbb{D}$.

Set $U_a := \{u \in D | T_a(u_2) < \zeta(u)\}$ and recal that $D_U := \{l \in D | \mathbf{p}(l) \in U\}$. Then it is well-known that $\mathbf{p}|_{D_{U_a}}$ and $\mathbf{p}|_{D_{U_a^c}}$ are independent. Moreover, $\mathbf{n}^\Psi[U_a] = \mathbf{n}^\Psi[T_a(u_2) < \zeta(u)] < \infty$ and then the first excursion of $\mathbf{p}|_{D_{U_a}}$ determines the hitting place $X(T_a^Y)$; more precisely, if we set

$$\kappa_a := \inf\{l > 0 | \mathbf{p}(l) \in U_a\} = \inf D_{U_a},$$

we have

$$\begin{aligned} T_a^Y - G_a^Y &= T_a(\mathbf{p}(\kappa_a)), \\ X(T_a^Y) - X(G_a^Y) &= \mathbf{p}(\kappa_a)_1(T_a(\mathbf{p}(\kappa_a))), \end{aligned}$$

$$\begin{aligned} G_a^Y &= \sum_{l \in (0,\kappa_a) \cap D_{U_a^c}} \zeta(\mathbf{p}(l)), \\ X(G_a^Y) &= \sum_{l \in (0,\kappa_a) \cap D_{U_a^c}} \mathbf{p}(\kappa_a)_1(\zeta(\mathbf{p}(l))). \end{aligned}$$

Note that $\mathbf{p}(\kappa_a)_1(\cdot)$ is the first component of the first excursion $\mathbf{p}(\kappa_a)$ in $\mathbf{p}|_{D_{U_a}}$.

By the standard argument concerning the Poisson point processes, we can deduce that $\left\{\mathbf{p}|_{D_{U_a^c}}, \kappa_a, \mathbf{p}(\kappa_a)\right\}$ are independent, so that $(T_a^Y - G_a^Y, X(T_a^Y) - X(G_a^Y))$ and $(G_a^Y, X(G_a^Y))$ are independent. The law of $\mathbf{p}(\kappa_a)$ is $\mathbf{n}^\Psi[\cdot; U_a]/\mathbf{n}^\Psi[U_a]$. Hence

$$E_{(0,0)}\left[e^{i\xi_1(X(T_a^Y)-X(G_a^Y))-\mu(T_a^Y-G_a^Y)}\right] = \frac{\mathbf{n}^\Psi[\exp(i\xi_1 u_1(T_a(u_2)) - \mu T_a(u_2)); U_a]}{\mathbf{n}^\Psi[U_a]}. \tag{2.11}$$

By the independence described above,

$$\begin{aligned} & E_{(0,0)}\left[e^{i\xi_1(X(T_a^Y)-X(G_a^Y))-\mu(T_a^Y-G_a^Y)}\right] \cdot E_{(0,0)}\left[e^{i\xi_1 X(G_a^Y)-\mu G_a^Y}\right] \\ = \; & E_{(0,0)}\left[e^{i\xi_1 X(T_a^Y)-\mu T_a^Y}\right] = \frac{u(a;\xi_1,\mu)}{u(0;\xi_1,\mu)}. \end{aligned} \tag{2.12}$$

Since $\{\mathbf{p}(l); l \in (0,\kappa_a) \cap D_{U_a^c}\}$ is a Poisson point process stopped at an independent exponential variable, we have

$$\begin{aligned} & E_{(0,0)}\left[e^{i\xi_1 X(G_a^Y)-\mu G_a^Y}\right] \\ = \; & \int_0^\infty dl\, \mathbf{n}^\Psi[U_a] e^{-l\mathbf{n}^\Psi[U_a]} \exp\left(-l\mathbf{n}^\Psi[1 - \exp(i\xi_1 u_1(\zeta(u)) - \mu\zeta(u)); U_a^c]\right) \\ = \; & \frac{\mathbf{n}^\Psi[U_a]}{\mathbf{n}^\Psi[U_a] + \mathbf{n}^\Psi[1 - \exp(i\xi_1 u_1(\zeta(u)) - \mu\zeta(u)); U_a^c]}. \end{aligned} \tag{2.13}$$

By the strong Markov property of $\mathbf{n}^\Psi$,

$$\begin{aligned} & \mathbf{n}^\Psi[\exp(i\xi_1 u_1(\zeta(u)) - \mu\zeta(u)); U_a] \\ = \; & \mathbf{n}^\Psi[\exp(i\xi_1 u_1(T_a(u_2)) - \mu T_a(u_2)); U_a] \cdot E_{(0,a)}\left[e^{i\xi_1 X(T_0^Y)-\mu T_0^Y}\right] \\ = \; & \mathbf{n}^\Psi[\exp(i\xi_1 u_1(T_a(u_2)) - \mu T_a(u_2)); U_a] \cdot \frac{u(-a;\xi_1,\mu)}{u(0;\xi_1,\mu)}. \end{aligned} \tag{2.14}$$

An elementary manipulation of these equalities yields

$$\mathbf{n}^\Psi[\exp(i\xi_1 u_1(T_a(u_2)) - \mu T_a(u_2)); U_a] = \frac{u(a;\xi_1,\mu)}{u(0;\xi_1,\mu)^2 - u(a;\xi_1,\mu)u(-a;\xi_1,\mu)}$$

among others. Then

$$\mathbf{n}^\Psi[U_a] = \lim_{\xi_1 \to 0, \mu \to +0} \mathbf{n}^\Psi[\exp(i\xi_1 u_1(T_a(u_2)) - \mu T_a(u_2)); U_a]$$

$$= \lim_{\mu\to+0} \frac{u(a;0,\mu)}{u(0;0,\mu)^2 - u(a;0,\mu)u(-a;0,\mu)}.$$

This quantity is concerned with the one-dimensional symmetric α-stable Lévy process $Y(t)$. Although we omit the further detail, $\mathbf{n}^\Psi[U_a]$ can be evaluated by the same method as Lemma 4.1 in [8]: $\mathbf{n}^\Psi[U_a] = \frac{\Psi(0,1)}{2h^{(\alpha)}(a)} = \frac{\Gamma(\alpha)\sin\frac{(\alpha-1)\pi}{2}\Psi(0,1)}{|a|^{\alpha-1}}$. □

3 Proof of Theorem 1.1

Let $\tau(a) = \inf\{t \geq 0 | Y(t) = 0, X(t) \geq a\}$ and $\sigma(a) = \inf\{t \geq 0 | \Xi(t) \geq a\}$ for $a \in \mathbb{R}$. Then $\sigma(a) = L_Y(\tau(a))$, $\Xi(\sigma(a)) = X(\tau(a))$, and $L_Y^{-1}(\sigma(a)) = \tau(a)$. Hence the first hitting time of interest, $\tau(a)$, can be studied via $\sigma(a)$ and its companions.

We now redefine the function $\varphi_\alpha(z;\mu_0,\mu_2)$. The coincidence of two definitions can be checked. Let $\mathbb{C}_+ = \{z \in \mathbb{C} | \Im z > 0\}$, $\overline{\mathbb{C}_+} = \{z \in \mathbb{C} | \Im z \geq 0\}$ and set

$$\varphi_\alpha(z;\mu_0,\mu_2) = \sqrt{\mu_0 + \Phi_\alpha(0,\mu_2)} \int_0^\infty dt\, E_{(0,0)}\left[e^{-\mu_0 t + iz\Xi(t) - \mu_2 L_Y^{-1}(t)}\right] \tag{3.1}$$

for $z \in \overline{\mathbb{C}_+}$ and $\mu_i \geq 0$ $(i = 0,2)$ such that $\mu_0 + \mu_2 > 0$. For $\mu_0 = \mu_2 = 0$, we set

$$\varphi_\alpha(z;0,0) = \frac{1}{\sqrt{\Phi_\alpha(1,0)}}(-iz)^{-(\alpha-1)/2} \quad \text{for } z \in \overline{\mathbb{C}_+} \setminus \{0\}, \tag{3.2}$$

where we employ the branch such that $1^{-(\alpha-1)/2} = 1$. For $\mu_i \geq 0$ $(i = 0,1,2)$ such that $\mu_0 + \mu_1 + \mu_2 > 0$, we define

$$I_\alpha(\mu_0,\mu_1,\mu_2) = \int_{-\infty}^\infty \frac{1}{2\pi(1+t^2)} \log(\mu_0 + \Phi_\alpha(\mu_1 t,\mu_2))dt, \tag{3.3}$$

convergence of which is verified using

$$0 \leq \Phi_\alpha(\xi_1,\mu_2) = |\xi_1|^{\alpha-1}\Phi_\alpha(1,|\xi_1|^{-\alpha}\mu_2) \sim \Phi_\alpha(1,0)|\xi_1|^{\alpha-1}, \tag{3.4}$$

as $|\xi_1|^\alpha/\mu_2 \to +\infty$.

If $\mu_0 = \mu_2 = 0$, it is elementary to verify $I_\alpha(0,\mu_1,0) = \log\left(\sqrt{\Phi_\alpha(1,0)}\mu_1^{(\alpha-1)/2}\right)$.

Proof of Theorem 1.1. We use the following in an crucial way:

- We use Theorem 1 in [3]: for any $z \in \overline{\mathbb{C}_+}$ and any $\theta \in \mathbb{R}$, it holds

$$\begin{aligned} &\sqrt{\mu_0 + \Phi_\alpha(0,\mu_2)}\varphi_\alpha(z;\mu_0,\mu_2) \\ &\quad = \exp\left(\int_0^\infty \frac{e^{-\mu_0 t}dt}{t} E\left[\left(e^{iz\Xi(t)} - 1\right)e^{-\mu_2 L_Y^{-1}(t)}; \Xi(t) > 0\right]\right), \end{aligned} \tag{3.5}$$

$$|\varphi_\alpha(\theta;\mu_0,\mu_2)|^2 = \varphi_\alpha(\theta;\mu_0,\mu_2)\varphi_\alpha(-\theta;\mu_0,\mu_2) = \frac{1}{\mu_0 + \Phi_\alpha(\theta,\mu_2)}. \tag{3.6}$$

- On the real line, we have

$$\varphi_\alpha(\theta;\mu_0,\mu_2) \sim \frac{\exp((\operatorname{sgn}\theta)\frac{\pi}{4}(\alpha-1)i)}{\sqrt{\Phi_\alpha(1,0)}|\theta|^{(\alpha-1)/2}} \quad \text{as } |\theta| \to \infty.$$

- On the positive imaginary axis, we have

$$\varphi_\alpha(i\mu_1;\mu_0,\mu_2) = \exp(-I_\alpha(\mu_0,\mu_1,\mu_2)).$$

- For any $a>0$ and $\mu_i \geq 0$ $(i=0,1,2)$ such that $\mu_0+\mu_1+\mu_2>0$,

$$\begin{aligned}
&1 - E_{(0,0)}\left[e^{-\mu_0\sigma(a)-\mu_1\Xi(\sigma(a))-\mu_2 L_Y^{-1}(\sigma(a))}\right] \\
&= \frac{1}{\varphi_\alpha(i\mu_1;\mu_0,\mu_2)}\int_{-\infty}^{\infty} d\theta \frac{1-e^{-ia(\theta-i\mu_1)}}{2\pi i(\theta-i\mu_1)}\varphi_\alpha(\theta;\mu_0,\mu_2).
\end{aligned}$$

We refer the reader to the forthcoming paper [6] for the detail of the proof. □

Remark 1 In the terminology of Chapter VI in [1], $\Xi(\sigma(a)) - a$ is the overshoot for a one-dimensional symmetric $(\alpha-1)$-stable Lévy process $\Xi(t)$. Adopting Exercise VI.1 and Lemma VIII.1 in [1], we have the following double Laplace transform:

$$\int_0^\infty da\, e^{-qa}\left(1 - E_{(0,0)}\left[e^{-\mu\Xi(\sigma(a))}\right]\right) = \frac{\mu^{(\alpha-1)/2}}{q(q+\mu)^{(\alpha-1)/2}}.$$

On the other hand, we set $\mu_0 = \mu_=0$ and take the Laplace transform of the both sides of Theorem 1.1(i) to obtain

$$\begin{aligned}
&\int_0^\infty da\, e^{-qa}\mu^{(\alpha-1)/2}\int_{-\infty}^{\infty} d\theta \frac{1-e^{-i(\theta-i\mu)a}}{2\pi i(\theta-i\mu)}\frac{1}{(-i\theta)^{(\alpha-1)/2}} \\
&= \mu^{(\alpha-1)/2}\int_{-\infty}^{\infty} d\theta \frac{\frac{1}{q}-\frac{1}{q+\mu+i\theta}}{2\pi i(\theta-i\mu)}\frac{1}{(-i\theta)^{(\alpha-1)/2}} \\
&= \frac{\mu^{(\alpha-1)/2}}{q}\int_{-\infty}^{\infty} d\theta \frac{1}{2\pi i(\theta-i(q+\mu))}\frac{1}{(-i\theta)^{(\alpha-1)/2}}.
\end{aligned}$$

The coincidence of these is verified by a simple application of the residue theorem.

4 The case for independent symmetric stable Lévy processes with different indices

Let $1<\alpha\leq 2$, $0<\beta\leq 2$, and $(X(t),Y(t))$ be such that $X(t)$ and $Y(t)$ are independent, $X(t)$ is symmetric β-stable, and $Y(t)$ is symmetric α-stable. In terms of the characteristic exponent, $\Psi(\xi_1,\xi_2) = |\xi_1|^\beta + |\xi_2|^\alpha$. When $(X(t),Y(t))$ is started from $(x_0,y_0)\in\mathbb{R}^2$, its law and expectation are denoted by $P_{(x_0,y_0)}$ and $E_{(x_0,y_0)}$, respectively. Let $L_Y(t)$ be the local time at 0 for $Y(\cdot)$: $L_Y(t) = \lim_{\varepsilon\to+0} \frac{1}{2\varepsilon}\int_0^t 1_{(-\varepsilon,\varepsilon)}(Y(s))ds$.

For $a \in \mathbb{R}$, we set $\tau(a) = \inf\{t \geq 0 | Y(t) = 0, X(t) \geq a\}$.

We define, for $z \in \mathbb{C}_+$, $\xi_1 \in \mathbb{R}$, and $\mu_i \geq 0$ $(i = 0,1,2)$ such that $\mu_0 + \mu_2 > 0$,

$$\begin{aligned}
\Phi_{\alpha,\beta}(\xi_1,\mu_2) &= 2\pi / \int_{\mathbb{R}} \frac{d\xi_2}{\mu_2 + |\xi_1|^\beta + |\xi_2|^\alpha}, \\
I_{\alpha,\beta}(\mu_0,\mu_1,\mu_2) &= \int_{-\infty}^{\infty} \frac{dt}{2\pi(t^2+1)} \log(\mu_0 + \Phi_{\alpha,\beta}(\mu_1 t,\mu_2)), \\
\varphi_{\alpha,\beta}(z;\mu_0,\mu_2) &= \exp\left(\frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{z}{t^2 - z^2} \log(\mu_0 + \Phi_{\alpha,\beta}(t,\mu_2))dt \right).
\end{aligned}$$

For $\mu_0 = \mu_2 = 0$, we define $I_{\alpha,\beta}(0,\mu_1,0) = \log\left(\sqrt{C_2(\alpha)}\mu_1^{\beta(\alpha-1)/(2\alpha)}\right)$ and $\varphi_{\alpha,\beta}(z;0,0) = \frac{1}{\sqrt{C_2(\alpha)}}(-iz)^{-\beta(\alpha-1)/(2\alpha)}$.

We obtain the following theorem by the same method as in §3. We refer the reader to the forthcoming paper [6] for the detail. □

Theorem 4.1 (i) *Let $a > 0$, $\mu_i \geq 0$ $(i = 0,1,2)$, and $\mu_0 + \mu_1 + \mu_2 > 0$.*
(i) *It holds*

$$\begin{aligned}
&E_{(-a,0)}\left[e^{-\mu_0 L_Y(\tau(0)) - \mu_1 X(\tau(0)) - \mu_2\tau(0)}\right] \\
&= e^{\mu_1 a} - e^{\mu_1 a}\exp(I_{\alpha,\beta}(\mu_0,\mu_1,\mu_2)) \int_{-\infty}^{\infty} d\theta \frac{1 - e^{-ia(\theta - i\mu_1)}}{2\pi i(\theta - i\mu_1)} \varphi_{\alpha,\beta}(\theta;\mu_0,\mu_2).
\end{aligned}$$

(ii) *As $s \to +0$ it holds*

$$\begin{aligned}
&1 - E_{(-a,0)}\left[e^{-\mu_0 s^{2(\alpha-1)} L_Y(\tau(0)) - \mu_1 s^{2\alpha/\beta} X(\tau(0)) - \mu_2 s^{2\alpha}\tau(0)}\right] \\
&\sim \frac{\exp(I_{\alpha,\beta}(\mu_0,\mu_1,\mu_2)) a^{\beta(\alpha-1)/(2\alpha)}}{\sqrt{C_2(\alpha)}\Gamma(1 + \frac{\beta(\alpha-1)}{2\alpha})} s^{\alpha-1},
\end{aligned}$$

where $\sim$ means that the ratio of the both sides converges to 1.

5 Some properties of modified resolvents

We modified the resolvents for $Y(t)$ in Section 2 and presented minimal(except Subsection 2.1) arguments for our application. The aim of this section is to characterize the modified resolvents in terms of the modified capacitary measure for $Y(t)$. Since the polarity of sets is determined solely by the process $Y(t)$, there is no addition to the classification results in our modification. We focus on the modified identities between some quantites in the potential theory for $Y(t)$. We do not need symmetry or (2.1) but state the results in terms of $\hat{P}_{(x_0,y_0)}$ and $\hat{E}_{(x_0,y_0)}$, the law and the expectation of the dual process, respectively.

In this section, we assume $(X(t), Y(t))$ is a Lévy process on $\mathbb{R}^2$ and employ the following notations for resolvents:

$$\begin{aligned} P_t^{\xi} f(y) &:= E_{(0,y)}\left[e^{i\xi X(t)} f(Y(t))\right], \\ U^{\xi,\mu} f(y) &:= E_{(0,y)}\left[\int_0^{\infty} e^{i\xi X(t)-\mu t} f(Y(t))dt\right] \end{aligned}$$

for $f(y) \in \mathcal{L}^{\infty}(\mathbb{R}) \cup \mathcal{L}^1(\mathbb{R})$. So we have $U^{\xi,\mu} f(y) = \int_{\mathbb{R}} f(y+z)U(dz;\xi,\mu)$, where $U(dy;\xi,\mu)$ is defined by (2.3) in Section 2. These quantities reduce, if $\xi = 0$, to $P_t f(y)$ and $U^{\mu} f(y)$ in [1, p.19,22], which employes q for μ. Our resolvents obey the same resolvent equation as the case $\xi = 0$:

Lemma 5.1 *Let $C_0 := \{f : \mathbb{R} \to \mathbb{R} | f$ is continuous and goes to 0 at infinity.$\}$.*

(i) *P_t^{ξ} maps C_0 into C_0; $\left(P_t^{\xi}\right)_{t \geq 0}$ forms a semigroup if $P_0^{\xi} = \mathrm{Id}$; not Markovian but satisfies $\|P_t^{\xi} f\| \leq \|f\|$; for each $f \in C_0$, $P_t^{\xi} f \to f$ uniformly as $t \to +0$.*

(ii) *For any $f(y) \in \mathcal{L}^{\infty}(\mathbb{R}) \cup \mathcal{L}^1(\mathbb{R})$, $\mu > 0$, and $\lambda > 0$, we have*

$$U^{\xi,\lambda} f(y) - U^{\xi,\mu} f(y) + (\lambda - \mu) U^{\xi,\lambda} U^{\xi,\mu} f(y) = 0. \tag{5.1}$$

(iii) *The range of $U^{\xi,\mu}$ does not depend on $\mu > 0$; we denote the range by $\mathcal{D}$; $\mu U^{\xi,\mu} f \to f$ uniformly as $\mu \to \infty$; $\mathcal{D} \subset C_0$ is a dense subspace; $U^{\xi,\mu}$: $C_0 \to \mathcal{D}$ is a bijection.*

Proof. (i) is a modified version of Proposition I.5 in [1, p.19]. (ii) can be checked by a standard argument. (iii) is shown by the same argument as in [1, p.23]. □

Obviously,

$$(\mu + \Psi(\xi, 0)) \int_{\mathbb{R}} U^{\xi,\mu} f(y)dy = \int_{\mathbb{R}} f(y)dy \tag{5.2}$$

for $f(y) \in \mathcal{L}^1(\mathbb{R})$. Set $T_B^Y = \inf\{t \geq 0 | Y(t) \in B\}$ and define the semigroup/resolvent with the killing upon entrance of B:

$$\begin{aligned} P_t^{B,\xi} f(y) &:= E_{(0,y)}\left[e^{i\xi X(t)} f(Y(t)); t < T_B^Y\right], \\ U_B^{\xi,\mu} f(y) &:= \int_0^{\infty} e^{-\mu t} P_t^{B,\xi} f(y)dt = E_{(0,y)}\left[\int_0^{T_B^Y} e^{i\xi X(t)-\mu t} f(Y(t))dt\right] \end{aligned}$$

These quantities reduce, if $\xi = 0$, to $P_t^B f(y)$ and $U_B^{\mu} f(y)$ in [1, p.47], which employes q for μ. Theorem II.5 in [1, p.47] is called 'Hunt's switching identity.' We also have an analog for the modified semigroup and resolvent.

Theorem 5.1 (modified Hunt's switching identity) *Let the modified dual semigroup $\hat{P}_t^{B,\xi}$ and the modified dual resolvent $\hat{U}_B^{\xi,\mu}$ be defined in the same way as $P_t^{B,\xi}$ and $U_B^{\xi,\mu}$, respectively, with $(X(t) - X(0), Y(t) - Y(0))$ replaced by $(X(0) - X(t), Y(0) - Y(t))$, i.e., the process travels in the opposite way.*

If either $f \in \mathcal{L}^\infty(\mathbb{R})$, $g \in \mathcal{L}^1(\mathbb{R})$ *or* $g \in \mathcal{L}^\infty(\mathbb{R})$, $f \in \mathcal{L}^1(\mathbb{R})$, *we have*

$$\begin{aligned} \int_{\mathbb{R}} dy\, g(y) P_t^{B,\xi} f(y) &= \int_{\mathbb{R}} dz\, f(z) \hat{P}_t^{B,-\xi} g(z), \\ \int_{\mathbb{R}} dy\, g(y) U_B^{\xi,\mu} f(y) &= \int_{\mathbb{R}} dz\, f(z) \hat{U}_B^{-\xi,\mu} g(z). \end{aligned}$$

To prove this theorem, we need two Lemmas. The first is a straightforward extension of Prop.II.1 in [1, p.44].

Lemma 5.2 *The following equality for two measures on* $\mathbb{R}^3 = \{(x,y,z)\}$ *holds.*

$$dy\, P_{(0,y)}\,[\,X(t) \in dx,\ Y(t) \in dz\,] = dz\, \hat{P}_{(0,z)}\,[\,-X(t) \in dx,\ Y(t) \in dy\,] \tag{5.3}$$

Proof. Let $f, g, h \in \mathcal{B}_+(\mathbb{R})$. We prove that the integrations of $g(y)f(z)h(x)$ by the two sides of (5.3) coincides.

$$\begin{aligned} \int_{\mathbb{R}} dy\, g(y) E_{(0,y)}\,[\,h(X(t)) f(Y(t))\,] &= \int_{\mathbb{R}} dy\, g(y) E_{(0,0)}\,[\,h(X(t)) f(y + Y(t))\,] \\ &= E_{(0,0)}\left[h(X(t)) \int_{\mathbb{R}} dy\, g(y) f(y + Y(t)) \right] \\ &= E_{(0,0)}\left[h(X(t)) \int_{\mathbb{R}} dz\, g(z - Y(t)) f(z) \right] \\ &= \int_{\mathbb{R}} dz\, f(z) E_{(0,0)}\,[\,h(X(t)) g(z - Y(t))\,] \\ &= \int_{\mathbb{R}} dz\, f(z) \hat{E}_{(0,0)}\,[\,h(-X(t)) g(z + Y(t))\,] \\ &= \int_{\mathbb{R}} dz\, f(z) \hat{E}_{(0,z)}\,[\,h(-X(t)) g(Y(t))\,]. \quad \square \end{aligned}$$

The second lemma is an extension of page 48, line 7 in [1].

Lemma 5.3 *If* $B \subset \mathbb{R}$ *is either open or closed,*

$$P_{(0,y)\to(x,z)}\left[\,t < T_B^Y\,\right] = \hat{P}_{(0,z)\to(-x,y)}\left[\,t < T_B^Y\,\right]. \tag{5.4}$$

Proof. By the same method as Corollary II.3 in [1, p.45], we can prove

$$\left((X_{(t-s)-}, Y_{(t-s)-};\, s \in [0,t]), P_{(0,y)\to(x,z)}\right) \overset{\text{law}}{=} \left((x + X_s, Y_s;\, s \in [0,t]), \hat{P}_{(0,z)\to(-x,y)}\right). \tag{5.5}$$

If B is open, it is clear that (see page 48, line 3 in [1])

$$\left\{t < T_B^{Y_s}\right\} \overset{pathwise}{=} \left\{t < T_B^{Y_{(t-s)-}}\right\} \tag{5.6}$$

and hence

$$P_{(0,y)\to(x,z)}\left[\,t < T_B^Y\,\right] \overset{(5.6)}{=} P_{(0,y)\to(x,z)}\left[\,t < T_B^{Y_{(t-s)-}}\,\right] \overset{(5.5)}{=} \hat{P}_{(0,z)\to(-x,y)}\left[\,t < T_B^Y\,\right].$$

If B is closed, we take a sequence of open sets $B_n \searrow B$ such that $\cap_n \overline{B_n} = B$. Then we have $T^Y_{B_n} \nearrow T^Y_B$ and $1\{t < T^Y_{B_n}\} \nearrow 1\{t < T^Y_B\}$ by Corollary I.8 in [1, p.22]. It is then elementary to observe

$$\begin{array}{ccc} P_{(0,y)\to(x,z)}\left[t < T^Y_{B_n}\right] & = & \hat{P}_{(0,z)\to(-x,y)}\left[t < T^Y_{B_n}\right] \\ \downarrow & & \downarrow \\ P_{(0,y)\to(x,z)}\left[t < T^Y_B\right] & & \hat{P}_{(0,z)\to(-x,y)}\left[t < T^Y_B\right] \end{array}$$

□

Proof of Theorem 5.1. We start with $f, g \in \mathcal{L}^\infty(\mathbb{R}) \cap \mathcal{L}^1(\mathbb{R})$.

By Lemma 5.3, the following functions are equal to each other.

$$g(y)f(z)e^{i\xi x}P_{(0,y)\to(x,z)}\left[t < T^Y_B\right] = g(y)f(z)e^{i\xi x}\hat{P}_{(0,z)\to(-x,y)}\left[t < T^Y_B\right]$$

We then integrate the both sides by the measures in the both sides of Lemma 5.2, respectively.

$$\begin{array}{ccc} \int_{\mathbb{R}} dy\, g(y)E_{(0,y)}\left[e^{i\xi X(t)}f(Y(t)); t < T^Y_B\right] & = & \int_{\mathbb{R}} dz\, f(z)\hat{E}_{(0,z)}\left[e^{-i\xi X(t)}g(Y(t)); t < T^Y_B\right] \\ \| & & \| \\ \int_{\mathbb{R}} dy\, g(y)P^{B,\xi}_t f(y) & & \int_{\mathbb{R}} dz\, f(z)\hat{P}^{B,-\xi}_t g(z) \end{array}$$

To loosen the condition $f, g \in \mathcal{L}^\infty(\mathbb{R}) \cap \mathcal{L}^1(\mathbb{R})$, we first set $\xi = 0$ to verify the both sides is absolutely convergent by using Fubini's theorem; next use truncation and the bounded convergence for any $\xi \in \mathbb{R}$. □

The capacitary measure is defined in [1], p.49. We define the modified capacitary measure for $B \subset \mathbb{R}$ which is either open or closed:

$$\mu^{\xi,\mu}_B(dz) := (\mu + \Psi(\xi, 0))\int_{\mathbb{R}} E_{(0,y)}\left[e^{i\xi X(T^Y_B) - \mu T^Y_B}; Y(T^Y_B) \in dz\right] dy.$$

Lemma 5.4 *For $f(y) \in \mathcal{L}^1(\mathbb{R})$,*

$$\int_{\mathbb{R}} f(y)dy = (\mu + \Psi(\xi, 0))\int_{\mathbb{R}} U^{\xi,\mu}_B f(y)dy + \int_{\mathbb{R}} U^{\xi,\mu}f(y)\mu^{\xi,\mu}_B(dy).$$

Proof. Use the strong Markov property at the instant T^Y_B. The version for $\xi = 0$ is the equation (1) in [1, p.51]. □

The next theorem is a modified version of Theorem II.7 in [1, p.50], which characterize the capacitary measure.

Theorem 5.2 *Define the measure $\mu^{\xi,\mu}_B U^{\xi,\mu}$ by $\int_{\mathbb{R}} f(z)\mu^{\xi,\mu}_B U^{\xi,\mu}(dz) = \int_{\mathbb{R}} U^{\xi,\mu}f(y)\mu^{\xi,\mu}_B(dy)$. Let $\xi \in \mathbb{R}$, $\mu > 0$ and suppose that B is either open or closed. Then*

$$\mu^{\xi,\mu}_B U^{\xi,\mu}(dz) = \hat{E}_{(0,z)}\left[e^{-i\xi X(T^Y_B) - \mu T^Y_B}\right] dz.$$

Moreover, $\mu^{\xi,\mu}_B$ is the unique $\mathbb{C}$-valued Radon measure on $\mathbb{R}$ satisfying the above equation.

Proof. Uniqueness follows from the denseness of $U^{\xi,\mu}f$ in C_0, see Lemma 5.1. By Lemma 5.4, we have

$$\int_{\mathbb{R}} f(z)\mu_B^{\xi,\mu}U^{\xi,\mu}(dz) = \int_{\mathbb{R}} f(y)dy - (\mu + \Psi(\xi,0)) \int_{\mathbb{R}} U_B^{\xi,\mu} f(y)dy.$$

We set $g \equiv 1 \in \mathcal{L}^\infty(\mathbb{R})$ and $f \in \mathcal{L}^1(\mathbb{R})$ in Theorem 5.2 to obtain the second term in the right side.

$$\begin{aligned}
&(\mu + \Psi(\xi,0)) \int_{\mathbb{R}} U_B^{\xi,\mu} f(y)dy \\
&= (\mu + \Psi(\xi,0)) \int_{\mathbb{R}} dz\, f(z)\hat{U}_B^{-\xi,\mu} 1_{\mathbb{R}}(z) \\
&= (\mu + \Psi(\xi,0)) \int_{\mathbb{R}} dz\, f(z) \int_0^\infty e^{-\mu t} \hat{P}_t^{B,-\xi} 1_{\mathbb{R}}(z)dt \\
&= (\mu + \Psi(\xi,0)) \int_{\mathbb{R}} dz\, f(z) \int_0^\infty e^{-\mu t} \hat{E}_{(0,z)} \left[e^{-i\xi X(t)}; t < T_B^Y \right] dt.
\end{aligned}$$

The first term in the right side is handled with (5.2) for the dual resolvent:

$$\begin{aligned}
\int_{\mathbb{R}} f(y)dy &= (\mu + \Psi(-(-\xi),0)) \int_{\mathbb{R}} \hat{U}^{-\xi,\mu} f(y)dy \\
&= (\mu + \Psi(\xi,0)) \int_{\mathbb{R}} dz\, f(z) \int_0^\infty e^{-\mu t} \hat{E}_{(0,z)} \left[e^{-i\xi X(t)} \right] dt.
\end{aligned}$$

Putting these together, we have

$$\begin{aligned}
&\int_{\mathbb{R}} f(z)\mu_B^{\xi,\mu}U^{\xi,\mu}(dz) \\
&= +(\mu + \Psi(\xi,0)) \int_{\mathbb{R}} dz\, f(z) \int_0^\infty e^{-\mu t} \hat{E}_{(0,z)} \left[e^{-i\xi X(t)} \right] dt \\
&\quad -(\mu + \Psi(\xi,0)) \int_{\mathbb{R}} dz\, f(z) \int_0^\infty e^{-\mu t} \hat{E}_{(0,z)} \left[e^{-i\xi X(t)}; t < T_B^Y \right] dt \\
&= (\mu + \Psi(\xi,0)) \int_{\mathbb{R}} dz\, f(z) \int_0^\infty e^{-\mu t} \hat{E}_{(0,z)} \left[e^{-i\xi X(t)}; t \geq T_B^Y \right] dt \\
&= (\mu + \Psi(\xi,0)) \int_{\mathbb{R}} dz\, f(z) \hat{E}_{(0,z)} \left[e^{-i\xi X(T_B^Y) - \mu T_B^Y} \right] \int_0^\infty e^{-\mu t} \hat{E}_{(0,0)} \left[e^{-i\xi X(t)} \right] dt \\
&= \int_{\mathbb{R}} dz\, f(z) \hat{E}_{(0,z)} \left[e^{-i\xi X(T_B^Y) - \mu T_B^Y} \right].
\end{aligned}$$

Since f is arbitrary integrable function, the proof is complete. □

References

[1] J. Bertoin, *Lévy processes*, Cambridge University Press, Cambridge, 1996.

[2] R. M. Blumenthal, R. K. Getoor, D. B. Ray, On the distribution of first hits for the symmetric stable processes, Trans. Amer. Math. Soc. **99** (1961) 540–554.

[3] Y. Isozaki, Asymptotic estimates for the distribution of additive functionals of Brownian motion by the Wiener–Hopf factorization method, J. Math. Kyoto Univ. **36** (1996) 211-227.

[4] Y. Isozaki, Fluctuation identities applied to the hitting time of a half-line in the plane, J. Theoret. Probab. **22** (2009) 57–81.

[5] Y. Isozaki, An asymptotic estimate for the hitting time of a half-line by two-dimensional Brownian motion, to appear in J. Math. Kyoto Univ. **49** no.3 (2009).

[6] Y. Isozaki, Hitting of a line or a half-line in the plane by two-dimensional symmetric stable Lévy processes, preprint, 2009. available at http://www.math.sci.osaka-u.ac.jp/~yasuki/

[7] S. C. Port, The first hitting distribution of a sphere for symmetric stable processes, Trans. Amer. Math. Soc. **135** (1969) 115–125.

[8] K. Yano, Y. Yano, M. Yor, On the laws of first hitting times of points for one-dimensional symmetric stable Lévy processes, to appear in Séminaire de Probabilités XLII, Lecture Notes in Math., Springer, Berlin, 2009. also available at http://www.springer.com/math/probability/book/978-3-642-01762-9